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ORIGINAL PAPER

Inscribed ball and enclosing box methods for the convex maximization problem

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Abstract Many important classes of decision models give rise to the problem of finding a global maximum of a convex function over a convex set. This problem is known also as concave minimization, concave programming or convex maximization. Such problems can have many local maxima, therefore finding the global maximum is a computationally difficult problem, since standard nonlinear programming procedures fail. In this article, we provide a very simple and practical approach to find the global solution of quadratic convex maximization problems over a polytope. A convex function achieves its global maximum at extreme points of the feasible domain. Since an inscribed ball does not contain any extreme points of the domain, we use the largest inscribed ball for an inner approximation while a minimal enclosing box is exploited for an outer approximation of the domain. The approach is based on the use of these approximations along with the standard local search algorithm and cutting plane techniques.

Keywords Local search algorithm · Non-convex optimization · Convex maximization · Inscribed ball

1 Introduction

In certain classes of nonlinear optimization problems, a local solution is a global one. For example, in minimization problems with a convex (or quasi-convex) objective

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function subject to convex constraints, the local minimum is a global solution. For non-convex functions there may be many local minima so that no local criterion will give information about the global minimum.

In this article, we consider the concave minimization problem, also called concave programming or convex maximization:

$$\begin{cases} \text{maximize} \;\; f(x), \\ \text{subject to } x \in D \end{cases} \tag{$\mathcal{P}$}$$

where $f : \mathcal{R}^n \to \mathcal{R}$ is a convex continuous function and D is a nonempty polyhedral set in $\mathcal{R}^n$ defined by

$$D = \{x \in \mathcal{R}^n \mid Ax \leq b\} = \{x \in \mathcal{R}^n \mid \langle a^i x\rangle \leq b_i, i = 1 \to m\}.$$

Assumption 1 *It is assumed that D is bounded (a polytope), full dimensional and there is no redundant constraint.*

We recall that a redundant constraint is a constraint that can be omitted from the system without changing the feasible domain.

Several problems from a variety of domains as technical design, economics, medicine, business and political science can be formulated as $(\mathcal{P})$. For example, the zero-one integer programming problem is equivalent to a convex quadratic maximization problem subject to linear constraints [17]. Many sub-problems of nonconvex problems lead to convex maximization; therefore, it turns out that convex maximization techniques also play an important role in other fields of global optimization [13].

A necessary optimality condition for a local solution y of $(\mathcal{P})$ is

$$\partial f(y) \cap N(D, y) \neq \emptyset. \tag{$\mathcal{LNC}$}$$

where $\partial f(\cdot)$, $N(\cdot,\cdot)$ stand respectively for the sub-differential and the normal cone defined by

$$N(D, y) = \{y^* \in \mathcal{R}^n \mid \langle y^*, x - y\rangle \leq 0 \;\; \forall x \in D\}.$$

Several interesting necessary and sufficient global optimality conditions characterizing a vector $z \in D$ satisfying

$$f(z) \geq f(x) \;\; \forall x \in D$$

have been developed [3,10,20,21].

Assume that z is not a minimum point of a convex function $f(\cdot)$, in other words $\exists x : f(x) < f(z)$ then a necessary and sufficient global optimality condition for $z \in D$ to be a global maximum for $(\mathcal{P})$ [21] is

$$\partial f(y) \cap N(D, y) \neq \emptyset \quad \text{for all } y \text{ such that } f(y) = f(z).$$

We note that for a differentiable $f(\cdot)$, due to $\partial f(y) = \{\nabla f(y)\}$, the latter conditions are equivalent to the following [20]:

$$\partial f(y) \subset N(D, y) \quad \text{for all } y \text{ such that } f(y) = f(z).$$

For the state-of-the-art in convex maximization including various algorithms and many applications, we refer to the textbooks [13,14], to the survey [1] and to the article [12].

It is worth noticing that in spite of being NP-hard, a local search for $(\mathcal{P})$ is relatively easy [5,19] due to the following method: from a given starting point $x^0 \in D$, at each iteration $(k = 0, 1, \ldots)$, x^{k+1} is a solution to the linear program over D with the objective function $\langle \nabla f(x^k)x \rangle$. The local search method can be written shortly like

$$x^{k+1} = argmax\{\langle \nabla f(x^k), x\rangle \mid x \in D\}. \tag{$\mathcal{LS}$}$$

Any accumulation point of the sequence $\{x^k\}$ generated by $(\mathcal{LS})$ satisfies the necessary local optimality conditions $(\mathcal{LNC})$.

An important property of convex functions is that every local and global maximum is attained at an extreme point of the feasible domain [18].

An obvious way to solve the convex maximization problem over a polytope is therefore a complete enumeration of the extreme points. Although most of the algorithms in the worst case will degenerate to complete inspection of all vertices of the polytope, this approach is computationally intractable for large problems [16].

We now consider the cutting plane method $(\mathcal{CPM})$ as described in [13]. Let a local (not global) maximum $y \in D$ be a vertex of the full dimensional polytope D. Let also $f(\cdot)$ be a strictly convex function. Following n edges at y, we find n vectors $y^1, y^2, \ldots, y^n$, which are the intersections of the edges with level set $\{x \mid f(x) = f(y)\}$. Then, a hyperplane $\{x \mid \langle c, x\rangle = \gamma\}$ containing these vectors can be constructed. In other words, one cuts off a part of D, where values of function $f(\cdot)$ are less or equal than $f(y)$. The same procedure is then applied to the remaining part of the feasible domain whenever this part is not empty. Although the concept is theoretically valid, it suffers in practice from the tailing off effect, i.e. cutting planes become closer or nearly parallel due to rounding errors so that they generate more and more local maxima. It remains a challenge in the global search to escape from local maxima.

In the remainder of the article, we consider $(\mathcal{P})$ with objective function $f(x) = \|x\|^2$, where $\|.\|$ stands for the Euclidean norm:

$$\begin{cases} \text{maximize } \ \|x\|^2, \\ \text{subject to } x \in D \end{cases} \tag{1}$$

The aim of this paper is to develop simple and quick methods that are able to find the global maximum with pre-defined accuracy. In Sect. 2 first, two main subproblems and their analytic solutions are defined, when D is a ball and a box respectively. Then, the section presents the largest inscribed ball and of the enclosing box of the feasible domain. Section 3 is devoted to necessary conditions and Sect. 4 to algorithms based on use of both approximation (ball and box). In Sect. 5, we present computational results and some ideas about how to enhance the outer approximation by using an hyper-rectangle [9] and the inner approximation by using an ellipsoid.

2 Preliminary results

2.1 The optima of two specific cases

In this subsection, we aim to determine optima of two special cases of Problem (1). The two subproblems are denoted (2) and (3) where the feasible domain is replaced by a ball, and respectively a box.

Let $w \in \mathcal{R}^n$ be a center of a ball, $r > 0$ be its radius, then a closed ball is defined by:

$$\{x \in \mathcal{R}^n \mid \|x - w\|^2 \leq r^2\}.$$

Lemma 1 *Consider the following problem:*

$$\begin{cases} \textit{maximize} \;\; \|x\|^2, \\ \textit{subject to}\; \|x - w\|^2 \leq r^2 \end{cases} \tag{2}$$

Vector $u = \left(1 + \frac{r^2}{\|w\|^2}\right) w$ *solves Problem (2).*

One can easily see that u calculated in this way solves the problem (2) due to KKT conditions.

Proof The KKT conditions imply that for the optimal solution u there is $\alpha \geq 1$ such that $u = \alpha w$. Since the global maximum of a convex function should be on the boundary of the ball, one has the following equation for α

$$\|x - w\|^2 = \|\alpha w - w\|^2 = r^2, \;\; \alpha \geq 1$$

which completes the proof. □

Let $L, U \in \mathcal{R}^n$ be two vectors such that $L_i < U_i, i = 1 \to n$. A box is defined by:

$$\left\{x \in \mathcal{R}^n \;:\; L_i \leq x_i \leq U_i, \;\; i = 1 \to n\right\}.$$

Lemma 2 *Consider the following problem:*

$$\begin{cases} \textit{maximize} \;\; \|x\|^2, \\ \textit{subject to}\; L_i \leq x_i \leq U_i, i = 1 \to n. \end{cases} \tag{3}$$

The global optimum is vector v such as:

$$v_i = \begin{cases} L_i, & if\; |L_i| > |U_i| \\ U_i, & if\; |L_i| \leq |U_i| \end{cases}, \;\; i = 1 \to n.$$

Since the problem (3) is separable, one can find its solution componentwise: v_i, given in Lemma 2, solves the following problem

$$\begin{cases} \text{maximize } \ x_i^2, \\ \text{subject to } L_i \leq x_i \leq U_i. \end{cases}$$

2.2 Inner and outer approximations

It is well known that inner and outer approximations of the feasible domain help to get bounds for the global optimum value. The largest ball inscribed into the polytope gives an inner approximation for the feasible domain D. Similarly, the minimum volume enclosing box of the polytope gives an outer approximation for the feasible domain D.

The optimal values over the inscribed balls S are strictly inferior to the global optimum value of Problem (1). On the one hand, as it is mentioned earlier, a global solution of convex maximization is always an extreme point, on the other hand, the largest inscribed ball of D does not contain any of its extreme points. Regarding enclosing boxes B, the upper bound can be an extreme point of D. Therefore, the global optimal value of Problem (1) is bounded as follows:

$$\max_{x \in S} f(x) < \max_{x \in D} f(x) \leq \max_{x \in B} f(x)$$

where S is the largest inscribed ball and B the enclosing box.

2.2.1 Ball center

Let $r(x)$ be the radius of the largest ball enclosed in D centered at an interior point $x \in int(D)$.

Let also $r_i(x)$ be the distance from a vector $x \in \mathcal{R}^n$ to a hyperplane $\{t : \langle a^i, t\rangle = b_i\}$ of the constraint i. Each distance is calculated analytically in the following way:

$$r_i(x) = \min_t\{\|x - t\| : \langle a^i, t\rangle = b_i\} = \frac{b_i - \langle a^i, x\rangle}{\|a^i\|}.$$

Therefore, $r(x)$ represents the minimum among these distances:

$$r(x) = \min\{r_i \mid i = 1 \to m\}.$$

W.l.o.g. it is assumed that $\|a^i\| = 1, \ \forall i$. Otherwise, one can normalize:

$$b_i = \frac{b_i}{\|a^i\|}, \ a^i = \frac{a^i}{\|a^i\|}, \ r_i(x) = b_i - \langle a^i, x\rangle.$$

From now

$$r(x) = \min\{b_i - \langle a^i, x\rangle \mid i = 1 \to m\}.$$

The radius for a given x satisfies

$$r(x) \le b_i - \langle a^i, x\rangle,\ i = 1 \to m.$$

We notice that $x \in D$ if and only if $r(x) \ge 0$.

The problem of finding the largest inscribed ball in D can be solved by unconstrained maximization of the concave function $r(x)$:

$$\begin{cases} \text{maximize } \ r(x), \\ \text{subject to } x \in \mathcal{R}^n. \end{cases}$$

In fact, the problem can be reformulated into a linear program in $\mathcal{R}^{n+1}$ also:

$$\begin{cases} \text{maximize } \ x_{n+1}, \\ \text{subject to } \langle a^i, x\rangle + x_{n+1} \le b_i \ : \ i = 1 \to m. \end{cases} \tag{4}$$

We denote by $S(s, r(s))$ the largest inscribed ball where s and $r(s)$ stand for optimal x and x_{n+1} of Problem (4) respectively.

Considering different norms, Hendrix et al. [8] focused on calculating the largest inscribed shapes, not only balls.

2.2.2 Outer box

In order to calculate the outer approximation, two vectors U and V are calculated so that theirs components are as an upper and a lower bound for each component x_i of $x \in D$.

For the feasible domain considered in this article, finding these vectors costs $2n$ linear programming problems, namely for each $i = 1 \to n$:

$$U_i = argmax\{\langle e^i, x\rangle : Ax \le b\},$$
$$L_i = argmin\{\langle e^i, x\rangle : Ax \le b\}$$

where $e^i = (0, \dots, 1, \dots, 0)^\top \in \mathcal{R}^n$.

3 Optimality condition checking

An evident way of global optimality condition checking is to compare a local solution over D with the global solution over its outer approximation.

Proposition 1 *Let y be a local solution of Problem* (1) *and let v be a global solution over the box B enclosing D. If $f(y) = f(v)$, then y is a global solution of Problem* (1).

Proof The inclusion $D \subset B$ implies: $f(v) \geq f(x),\ x \in D$. When $f(v) = f(y)$, the inequality $f(y) \geq f(x)\ \forall x \in D$ holds. Therefore y is a global optimum of (1). □

For a differentiable convex function f the necessary global optimality condition can be written like [19,20]

$$z = argmax(1) \Rightarrow \langle \nabla f(y), x - y \rangle \leq 0, \forall x \in D, \forall y : f(y) = f(z).$$

Now we describe another possibility of necessary optimality condition checking which is helpful for escaping from local optima. A constraint j is inactive for vector $y \in D$ if $\langle a^j, y \rangle < b_j$. For an inactive constraint j, the following convex problem is considered:

$$\begin{cases} \text{maximize } \langle a^j, x \rangle, \\ \text{subject to } f(x) \leq f(y), x \in D_j, \end{cases} \tag{$\mathcal{P}_j$}$$

where $a^j \in \mathcal{R}^n$ is the normal vector of j-th constraint and D_j defined by

$$D_j = \{x \mid \langle a^k, x \rangle \leq b_k; \text{for all } k \neq j\}.$$

Proposition 2 ([7]) *Let* $w^j = argmax(\mathcal{P}_j)$. *If* $\langle a^j, w^j \rangle < b_j$, *and* $f(w^j) = f(y)$ *then* y *is not the global solution to Problem* (1).

Proof The full dimensionality from Assumption 1 implies the regularity, see page 284 in [2]. By first order optimality conditions, there exist

$$\lambda \geq 0, \mu_k \geq 0, k = 1 \rightarrow m, k \neq j$$

such that

$$\begin{cases} a^j - \lambda f(w^j) - \sum_{k \neq j} \mu_k a^k = 0 \\ \lambda(f(w^j) - f(y)) = 0 \\ \mu_k \left(\langle a^k, w^j \rangle - b_k \right) = 0. \end{cases}$$

Again due to Assumption 1, there exists $x \in D$ such that $\langle a^j, x \rangle = b_j$. Hence, for any $x \in D$ satisfying $\langle a^j, x \rangle = b_j$, the inequality $\langle a^j, w^j \rangle < b_j$ implies

$$0 < \langle a^j, x - w^j \rangle = \lambda \langle \nabla f(w^j), x - w^j \rangle + \sum_{k \neq j} \mu_k \langle a^k, x - w^j \rangle.$$

It can be shown that for $x \in D_j$

$$\sum_{k \neq j} \mu_k \langle a^k, x - w^j \rangle \leq \sum_{k \neq j} \mu_k (b_k - \langle a^k, w^j \rangle) = 0,$$

and therefore

$$0 < \langle a^j, x - w^j \rangle \le \lambda \langle \nabla f(w^j), x - w^j \rangle.$$

Since $f(\cdot)$ is convex, one has

$$0 < \langle \nabla f(w^j), x - w^j \rangle \le f(x) - f(w^j) = f(x) - f(y),$$

which proves the Proposition $\exists x \in D : f(x) > f(y)$. □

The cutting-plane method is also useful in optimality conditions checking. A description of the cutting-plane method is presented in Algorithm 1.

Algorithm 1: Cutting-plane method ($\mathcal{CPM}$).

Data: D^{k-1}, y^k
Result: D^k

- Let a local maximum y^k be a vertex of the full dimensional polytope D^{k-1}, the domain after the $(k-1)$-th cutting-plane with $D^0 = D$.
- Following n edges at y^k, find n vectors $y^{k_1}, y^{k_2}, \ldots, y^{k_n}$, which are the intersections of the edges with level set $\{x \mid f(x) = f(y^k)\}$.
- Let c be the normal of the hyperplane formed by the set of vectors $\{y^{k_1}, y^{k_2}, \ldots, y^{k_n}\}$ such as $\langle c, y^{k_i} \rangle > \langle c, y^k \rangle$.
- $D^k = D^{k-1} \cap \{x \in \mathcal{R}^n \mid \langle c, x \rangle \ge \langle c, y^{k_i} \rangle\}$

Proposition 3 *Let $\langle c, x \rangle = \gamma$ be the cutting-plane equation obtained from local maximum y such that $\gamma > \langle c, y \rangle$. Let $\hat{x}$ be the solution of the following problem:*

$$\begin{cases} \text{maximize} \quad \langle c, x \rangle, \\ \text{subject to } x \in D \end{cases} \tag{5}$$

Then y is a global optimum of Problem $(\mathcal{P})$ if $\langle c, \hat{x} \rangle < \gamma$.

Proof Let $\langle c, \hat{x} \rangle = \hat{b}$. The new constraint (cutting-plane) is $\langle c, x \rangle \ge \gamma$ and let $\mathcal{L}_f(\alpha)$ be the Lebesgue set of f on α:

$$\mathcal{L}_f(\alpha) = \{x \mid f(x) \le \alpha\}.$$

If $\gamma < \hat{b}$ then there exists $x \in \mathcal{R}^n$ such that $\hat{b} \ge \langle c, x \rangle \ge \gamma$. Moreover, $\forall x \in D$, $\langle x, c \rangle \le \hat{b}$, so there is $x \in D$ such that $\hat{b} \ge \langle c, x \rangle \ge \gamma$. Thus, the domain is nonempty.

If $\gamma > \hat{b}$ then $\forall x \in \mathcal{R}^n$, $\langle c, x \rangle \le \hat{b} \le \gamma$. Moreover, $\forall x \in D$, $\langle x, c \rangle \le \hat{b}$. Thus, there is no $x \in D$ such that $\langle c, x \rangle \ge \gamma$, the domain becomes empty after adding the cutting-plane. Let us recall a cutting-plane property:

$$\{x \in D \mid \langle c, x \rangle \le \gamma\} \subset \mathcal{L}_f(f(y)).$$

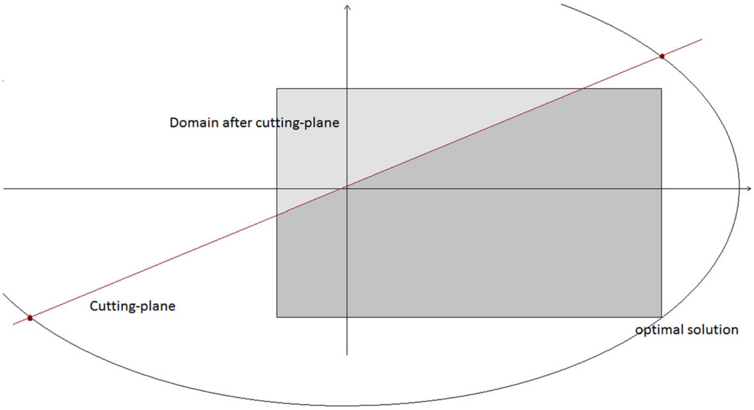

Fig. 1 Domain after a cutting-plane

By the definition of the Lebesgue set, it implies that $\forall x \in D$, $f(x) \leq f(y)$, thus y is a global maximum over D. □

Of course, there are some cases where the domain remains non-empty after a cutting-plane even y is a global maximum, see Fig. 1. We summarize the global optimality condition checking thanks to cutting-plane as follows:

1. If there exists a vector $y^{k_i} \in D^{k-1}$, then the domain D^k is non-empty.
2. If each vector $y^{k_i} \notin D^{k-1}$, $i = 1 \to n$, and D^k is not empty then y^k is undefined.
3. If each vector $y^{k_i} \notin D^{k-1}$, $i = 1 \to n$ and D^k is empty then y^k is the global optimum over D^{k-1}.

4 Algorithms

4.1 Inner approximation

The inner approximation is based on use of the largest inscribed ball.

The main idea is:

- while the radius of the inscribed ball is greater than a threshold value:
  - construct the largest inscribed ball in the domain by solving the linear programming problem (4);
    let w be the ball center;
  - solve Problem (2); *let u be a solution of Problem* (2);
  - add an hyperplan tangent at u as a new temporary constraint to Problem (1): $\langle w, x\rangle \geq \langle w, u\rangle$ or equivalently $\langle w, x\rangle \geq \|w\|^2 + r^2$ due to *lemma 1*;
- run local search ($\mathcal{LS}$) from the current vector u;
  let y be the local optimum (stationary point);
- construct cutting-plane from y.

Algorithm 2: Inner approximation: IA

Data: $Ax \leq b$
Result: Global optimum of Problem (1)
Initialization : $r_{min} = \epsilon$ and $condition = false$;
while $condition = false$ **do**
  while $r > r_{min}$ **do**
    Find r, w the solution of Problem (4);
    Find u the global optimum of Problem (2);
    Add a temporary constraint to Problem (1): $\langle w, x\rangle \geq \|w\|^2 + r^2$;
  end
  Remove all temporary constraints;
  Find local maximum y by ($\mathcal{LS}$) from the current u;
  Construct cutting plane from y by ($\mathcal{CPM}$);
  / Prop. 2 or/and Prop. 3 */*
  if *an optimality condition is satisfied* **then**
    $condition = true$;
    Keep in memory the optimum;
  else
    Add the cutting-plane constraint $\langle c, x\rangle \geq \left\langle c, y^{k_i}\right\rangle$ to Problem (1);
  end
end
Return the global maximum;

4.2 Enhancement with outer approximation

Let v be the maximum of Problem (3) over enclosing box B. If the vector v belongs to D, then obviously v is a global maximum to $(\mathcal{P})$ too. Assume that $v \notin D$.

The main idea of this enhancement is to combine both inner and outer approximations to find a better starting point for local search ($\mathcal{LS}$).

Let u be the maximum of Problem (2) over inscribed ball S. Since $u \in D$ and $v \notin D$, taking a convex combination of u and v one finds an intersection with the boundary of D, which is a good starting point for a local search.

First, $v \notin D$ implies that there are indices j such that $\left\langle a^j, v\right\rangle > b_j$ and we denote a set of such indices of violated constraints by

$$J = \{j \mid \langle a^j, v\rangle > b_j\}.$$

The intersection points are calculated directly from the following equation

$$\langle a^j, u + \alpha_j(v - u)\rangle = b_j$$

in $\alpha_j : 0 \leq \alpha_j \leq 1$ for $j \in J$. Then, the nearest one to u is determined by

$$\alpha = \min\{\alpha_j \mid 0 \leq \alpha_j \leq 1, \quad j \in J\}.$$

Algorithm 3: Inner and outer approximation: IOA

Data: $Ax \leq b$
Result: Global optimum of Problem (1)
Initialization : $condition = false$;
while $condition = false$ **do**
  Resolve Problem (4);
  Find u the global optimum of Problem (2);
  Find v the global optimum of Problem (3);
  Find the intersection of segment $[uv]$ with boundary of D;
  Find local maximum y by $(\mathcal{LS})$ from the intersection point;
  Construct cutting plane from y by $(\mathcal{CPM})$;
  / Prop. 1 or/and Prop. 2 or/and Prop. 3 */*
  if *an optimality condition is satisfied* **then**
    $condition = true$;
    Keep in memory the optimum;
  else
    Add the cutting-plane constraint $\langle c, x\rangle \geq \left\langle c, y^{k_i}\right\rangle$ to Problem (1);
  end
end
Return the global maximum;

5 Computational experiments and future works

5.1 Computational experiments

The IA and IOA algorithms have been run on a selection of test instances of convex maximization problems taken from "A collection of test problems for constrained global optimization algorithms" (noted by TP# the chapter of the instance) [6] and "An algorithm for maximizing a convex function over a simple set" (noted by P# the number of the instance) [4].

The algorithms are implemented on Scilab (http://www.scilab.org/), linear programs are solved using Linpro solver (from add-on Quapro). The tests are performed on a computer with a IntelCore2 Duo processor, 3.16 GHz CPU and 4 GB of RAM.

The numerical results are presented in the table below and the meanings for all columns in the table are as follows:

- n stands for number of variables;
- *iIA* stands for number of local searches for IA to solve Problem (1);
- *iIOA* stands for number of local searches for IOA to solve Problem (1);
- *bkv* stands for the best known value;
- the global optimal value obtained by the algorithms developed in this paper;
- *tIA* stands for the average computing time for IA in seconds (with $r_{min} = 0.01$);
- *tIOA* stands for the average computing time for IOA in seconds.

Figure 2 illustrates a sequence of the inscribed balls obtained by the IA algorithm, which show visually the convergence to the global solution for test problem P4. Our

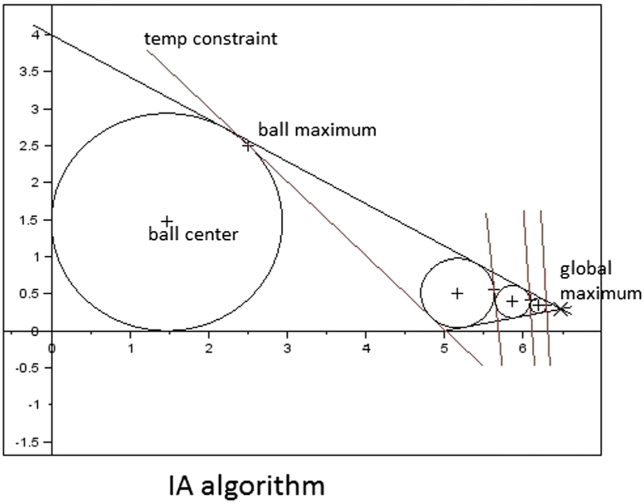

Fig. 2 One iteration of IA algorithm

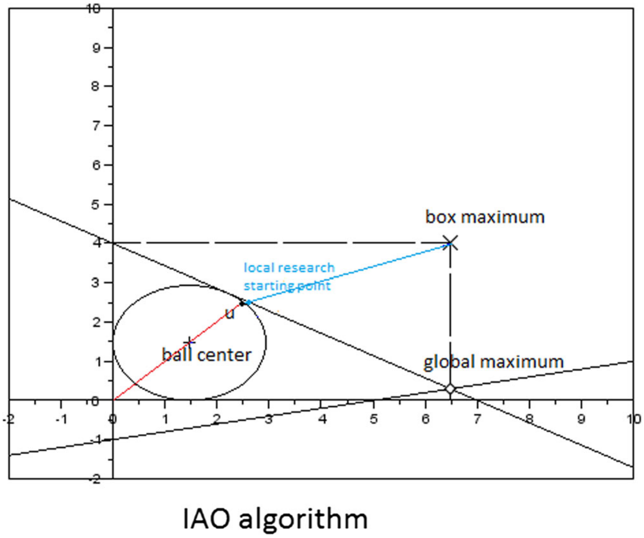

Fig. 3 One iteration of IOA algorithm

local search method requires an initial starting point, and one can see in Fig. 3, the calculation of the starting point for the IAO algorithm.

We should warn the reader that although both algorithms found the global optimal solutions for all test problems (see Table 1) within short computing time (less than 1 s for $n \leq 10$, less than 2 s for $n \leq 50$ and 35 s. for $n = 200$). Computing times are not relevant, and do not challenge well-known algorithms, our main aim is quality of solutions obtained at each iteration, and to reach an optimal value in few iterations.

Table 1 Computational results for benchmarks from [4,6]

Problem	n	iIA	iIOA	bkv	Optimal value	tIA	tIOA
TP2.1	5	3	3	−17.0000	−17.0000	0.1	0.2
TP2.6	10	2	4	−39.0000	−39.0000	0.3	0.7
TP2.7.1	20	3	2	−394.7506	−394.7506	1.5	1.0
TP2.7.3	20	3	2	−8695.01193	−8695.01193	1.5	1.0
P4	2	1	1	42.0976	42.0976	0.1	0.1
P6	2	2	1	162.0000	162.0000	0.1	0.1
P11	100	2	1	1541089.	1541089.	6.0	2.5
P11	200	2	1	4150.4101	4150.4101	35.0	14.7

Table 2 Approximation results for benchmarks from [4,6]

Problem	fIA	fIOA	bIA	bIOA
TP2.1	0.68	1.42	0.92	1.42
TP2.6	0.60	1.08	0.81	1
TP2.7.1	0.35	>10	0.95	1.54
TP2.7.3	0.74	1.82	0.93	1.31
P4	0.86	1.38	0.86	1.38
P6	0.55	1	0.81	1
P11	0.82	1	0.83	1
P11	0.83	1	0.86	1

We now turn our attention to Table 2 that illustrates the ratio of the best known value to the optimal values of the objective function over the approximations (box and ball) for the IA and the IOA algorithms.

The meanings for columns in the table are as follows, where z is a global optimum of Problem (1), u is the global optimum of a ball and v is the global optimum of a box:

- *fIA* stands for the best value of $\frac{f(u)}{f(z)}$ during the first iteration of IA algorithm;
- *fIOA* stands for the best value of $\frac{f(v)}{f(z)}$ during the first iteration of IOA algorithm;
- *bIA* stands for the best value of $\frac{f(u)}{f(z)}$;
- *fIOA* stands for the best value of $\frac{f(v)}{f(z)}$;

On average, for a larger benchmark of test problems, a ball optimum and a box optimum are between 70–140 % of the best known value for the first iteration and between 90–115 % for the last iteration. The disparity between both approximations comes from the radius threshold for inscribed balls and the shape of the feasible domain D.

Computing results reinforce the idea of seeking the global optimum from approximations of the feasible domain. However, as balls and boxes may not fit well into the feasible domain, the following subsections present some better approximations for a random polyhedron.

5.2 From box to hyper-rectangle

The outer approximation by a box could be non representative to the feasible domain. Indeed, the box is constructed by considering the lower and upper bound in each dimension, and it does not take into account the shape of the feasible domain. In order to approximate the feasible domain in a better way, one could try to find an enclosing minimum volume hyper-rectangle. Let a matrix $\mathcal{K}$ be defined as follows:

$$\mathcal{K} = (d^1, \ldots, d^n) \quad with\ \langle d^i, d^j \rangle = 0,\ i \neq j\ and\ det(\mathcal{K}) \neq 0.$$

So, a problem of the smallest hyper-rectangle enclosing the polytope D is defined as follows:

$$\begin{cases} \underset{d^i}{minimize}\ \prod_{i=1}^{n} \left\langle d^i, U_i - L_i \right\rangle, \\ \text{subject to } U_i = argmax \left\{ \left\langle d^i, x \right\rangle \ :\ Ax \leq b \right\}, \\ \qquad\qquad\quad L_i = argmin \left\{ \left\langle d^i, x \right\rangle \ :\ Ax \leq b \right\},\ i = 1 \rightarrow n. \end{cases}$$

There is infinite number of orthogonal bases and finding the best one is not an easy task. This problem is of the same degree of difficulty as (1). Therefore, the problem is taken with another point of view. The goal is to construct a hyper-rectangle according to the shape of the feasible domain. In order to calculate $\mathcal{K}$, the Gram–Schmidt method [11] is applied, see Algorithm 4 and Fig. 4. First, we choose $u^1 = v^1 = d^1$. At the very beginning, d^1 is chosen from the active edges at the local optimum y and $v^2, \ldots, v^n = e^2, \ldots, e^n$. Then, $\mathcal{K}$ is formed by the rotation of the global basis which transform e^1 into $\dfrac{d^1}{\|d^1\|}$. We notice that the approximation's quality depends on choice of the starting edge. Moreover, this process doesn't guarantee the minimality of volume for the hyper-rectangle.

Algorithm 4: Gram–Schmidt method.

Let $proj_u(v)$ be the projection operator of v on u by

$$proj_u(v) = \frac{\langle u, v \rangle}{\langle u, u \rangle} u.$$

Let $v^1, \ldots, v^n$ be n vectors, we define the recurrence: $u^1 = v^1$, $u^2 = v^2 - proj_{u^1}(v^2)$ and

$$u^k = v^k - \Sigma_{i=1}^{k-1} proj_{u^j}(v^k).$$

The basis formed by the set of vectors $\{v^1, \ldots, v^n\}$ is orthonormal.

Lemma 3 *The global optimum z over the hyper-rectangle can be calculated by:*

$$z_i = \begin{cases} |\langle d^i, L_i \rangle|, & if\ |\langle d^i, L_i \rangle| > |\langle d^i, U_i \rangle| \\ |\langle d^i, U_i \rangle|, & if\ |\langle d^i, L_i \rangle| \leq |\langle d^i, U_i \rangle| \end{cases} \ i = 1 \rightarrow n.$$

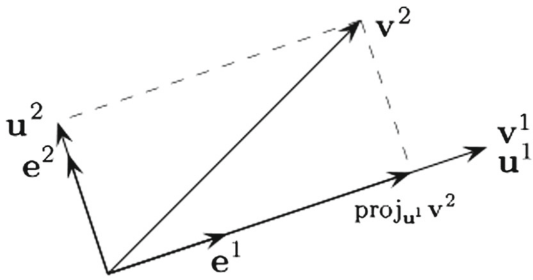

Fig. 4 Gram–Schmidt process where v^1 is colinear to e^1

Proof The proof is similar to the case of the box. □

5.3 From ball to ellipsoid

Naturally, further improvement of the approach can be done by using the largest inscribed ellipsoid instead of the ball since the former approximates the feasible domain better than the latter.

Finding the largest inscribed ellipsoid is a convex optimization problem [15], while the largest inscribed ball problem is solved by a linear programming [8].

6 Concluding remarks

In this article, we have developed algorithms for finding the global solution for quadratic convex maximization problem. The global optimal solutions are found for all test problems considered in few local searches.

Making a better choice of d^1 in Sect. 5.2 is a challenging issue and using the inscribed ellipsoid could improve the inner approximation. We keep these interesting topics for our near future researches.

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