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Largest Inscribed Ball and Minimal Enclosing Box for Convex Maximization Problems

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Abstract Many important classes of decision models give rise to the problem of finding a global minimum of a concave function over a convex set. Since such a function may have many local minima, finding the global minimum is a computationally difficult problem, where standard nonlinear programming procedures fail. The two proposed methods are simple and quick, using the largest inscribed ball and the minimal enclosing box as approximation for cutting-plane method.

Keywords: convex maximization, ball center

1. Introduction

In certain classes of nonlinear problems the local solution is always the global one. For example, in minimization problems with a convex (or quasi-convex) objective function subject to convex constraints the local minimum is the global solution. For non-convex functions there may be many local minima so that no local criteria will give information about the global minimum.

In this article, we consider the non-convex optimization problem, also known as concave minimization, concave programming or convex maximization:

$$\begin{cases} \text{maximize } \ f(x), \\ \text{subject to } x \in D \end{cases} \tag{1}$$

where $f : \mathcal{R}^n \to \mathcal{R}$ is a convex continuous function and D is a nonempty, convex compact in $\mathcal{R}^n$, a polytope defined by

$$D = \{x \in \mathcal{R}^n \mid Ax \leq b\} = \{x \in \mathcal{R}^n \mid \langle a^i, x\rangle \leq b_i,\ i = 1, \ldots, m\}.$$

For the state-of-the-art in convex maximization including various algorithms and abundant applications, we refer to the textbooks [9, 10] and to survey [1].

An important property of convex functions is that every local and global maximum is achieved at some extreme point of the feasible domain [13]. Several interesting necessary and sufficient global optimality conditions characterizing a point $z \in D$ satisfying $f(z) \geq f(x),\ \forall x \in D$ have been proposed [2, 6, 7, 15, 16].

An obvious way to solve the concave programming problem is a complete enumeration of the extreme points. Although most of the algorithms in the worst case will degenerate to complete inspection of all vertices of the polytope, this approach is computationally infeasible for large problems [12].

In this article, algorithms are specialized for solving the following problem, maximization of distance to the origin, a quadratic problem:

$$\begin{cases} \text{maximize } \ \|x\|^2, \\ \text{subject to } x \in D \end{cases} \tag{2}$$

where D is a full dimensional polytope.

2. Optimization methods

Firstly, let us concentrate on the local search and the cutting plane method. A local search with starting point x for (1) is relatively easy due to the method [4, 14]:

$$x^{k+1} = argmax\{\langle \nabla f(x^k), x\rangle \mid x \in D\}.$$

When $x^{k+1} = x^k$, then x^k is a local maximum of D.

Let a local maximum $y \in D$ be a vertex of the full dimensional polytope D. Following n edges at y, we find n points $y^1, y^2, ..., y^n$, which are the intersections of the edges with level set $\{x \mid f(x) = f(y)\}$. Then, hyper-plane $\{x \mid \langle c, x\rangle = \gamma\}$ that contain the points are built [9].

By the convexity of the objective function $f(\cdot)$ problem (2) is equivalent to:

$$\begin{cases} \text{maximize } \|x\|^2, \\ \text{subject to } x \in D, \ \ \langle c, x\rangle \geq \gamma \end{cases}$$

In other words, one cuts off a part of D, where values of function $f(\cdot)$ are less or equal than $f(y)$. The same procedure is then applied to the remaining part of the feasible set whenever this part is not empty.

However, despite such nice theoretical idea, this approach suffers, in practice, from the tailing off effect, i.e. cutting planes become closer or nearly parallel due to rounding errors so that they generate more and more local maxima. It remains the challenge in global search step: how to escape from a local maximum area?

Proposed methods are based on two sub-problems. The first one is the maximum value of (2) where D is a ball, noted. The second one is the maximum value of (2) where D is a box.

Lemma 1.

$$\begin{cases} \textit{maximize } \|x\|^2, \\ \textit{subject to } \|x - w\|^2 \leq r^2 \end{cases} \tag{3}$$

The optimal solution is $u = \left(1 + \frac{r}{\|w\|}\right) w$.

The largest ball inscribed into the polytope D is based on Murty et al. research works [11]. The radius of an inner ball is the minimal distance from its center x to the constraints of the feasibility domain. The center of the largest ball inscribed into D is the solution of the maximum value of the minimum radius of a point $x \in D$, with $\|a^i\| = 1$:

$$\begin{cases} \text{maximize } x_{n+1}, \\ \text{subject to } \left\langle a^i, x\right\rangle + x_{n+1} \leq b_i, \, i = 1, \ldots, n \end{cases} \tag{4}$$

The second sub-problem is the maximum value of (2) where D is a box.

Lemma 2.

$$\begin{cases} \textit{maximize } \|x\|^2, \\ \textit{subject to } L_i \leq x_i \leq U_i, i = 1, \ldots, n \end{cases} \tag{5}$$

The global optimum is $v = \left(\ max\{\mid L_i \mid, \mid U_i \mid\} \ \right)$, $i = 1, \ldots, n$.

In order to calculate the outer approximation, let U and L be the upper and lower bounds for each dimension for D. The domain is convex, so for all $i = 1, \ldots, n:$

$$U_i = argmax\left\{\left\langle e^i, x\right\rangle \ : \ Ax \leq b\right\}, \, L_i = argmin\left\{\left\langle e^i, x\right\rangle \ : \ Ax \leq b\right\}, \, e^i = (0, \ldots, 1, \ldots, 0)^T$$

Both methods have the same process. One method uses an inner approximation (IA) with the largest inscribed ball. The second one is based on an inner and outer approximation (IOA) using largest inscribed ball and minimal enclosed box. We expose the main step of the two methods:

1. Find a candidate: IA resolves (4) then (3). IOA resolves (4) then (3), and (5).
2. Find a local maximum from the candidate.
3. Use cutting-plane until the domain is empty.
4. Return global maximum.

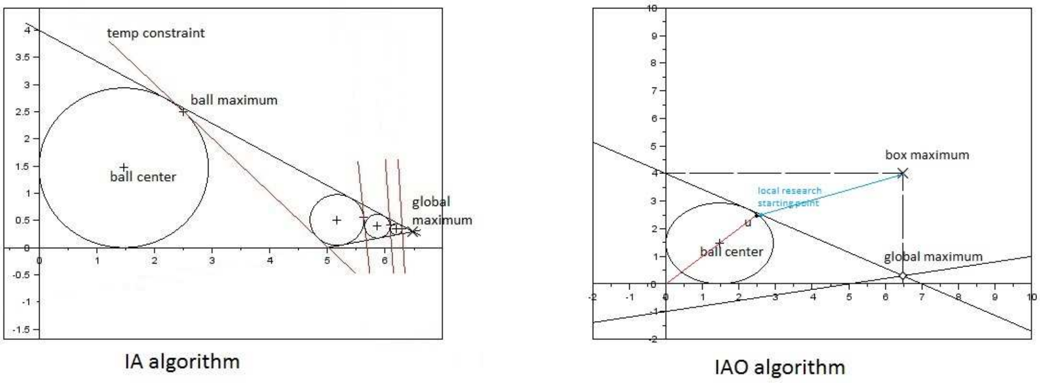

Figure 1: One iteration of IA and IOA algorithm.

3. Results and discussions

The IA and IAO algorithms have been tested on a bunch of convex maximization problems taken from "A collection of test problems for constrained global optimization algorithms" [5] (noted: TP plus the chapter of the test) and "An algorithm for maximizing a convex function over a simple set" [3] (noted: P plus the number of the test).

We present the numerical results in the table below and the meanings for all columns in the table follow: number of variables; number of local searches for IA; number of local searches for IOA; the best value found; the global optimal known value; the average computing time for IA in seconds; the average computing time for IOA in seconds.

Problem	n	IA LS	IOA LS	the best value	optimal value	time IA	time IOA
TP2.1	5	3	3	-17	-17	0.1	0.2
TP2.6	10	2	4	-39	-39	0.3	0.7
TP2.7.1	20	3	2	-394.7506	-394.7506	1.5	1.0
TP2.7.3	20	3	2	-8695.01193	-8695.01193	1.5	1.0
P4	2	1	1	42.0976	42.0976	0.1	0.1
P6	2	2	1	162	162	0.1	0.1
P11	100	2	1	1541089	1541089	6	2.5
P11	200	2	1	4150.41013	4150.41013	35	14.7

The best known solutions are found for all test problems considered in few local searches. Average computing time is calculated in the case of all conditions are checked at each iteration (full dimensional, active constraints, normal cone, etc.).

4. Conclusion

In this article, in order to find the global solution of a quadratic convex maximization problem, two algorithms are described. They are based on using the largest inscribed ball and the minimal enclosing box as an approximation for cutting-plane method. These methods are simple, quick and provide the global optimum.

Currently, we use box built on the orthonormal basis. In future work, a box with an other orthonormal set will be used (also named cuboid) thanks to Gram-Schmidt algorithm [8] in order to build the minimal box enclosing the domain D.

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